

Quantum Coherence and Holevo Bound

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We establish the equivalence between the loss of coherence due to mixing in a quantum system and the loss of information after performing a projective measurement. Subsequently, it is demonstrated that the quantum discord, a measure of correlation for the bipartite system $\rho_{Alice \leftarrow Bob}$, is identical to the minimum difference (over all projectors $\{|i\rangle\langle i|\}$) between local coherence (LQICC monotone) on Bob side and coherence of the reduced density matrix ρ^B .

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I. INTRODUCTION

Coherence is one of the basic features of quantum mechanics that distinguishes it from classical physics. Preserving and monitoring coherence are the fundamental challenges to be overcome for the physical applications of quantum principles. Although it is fundamental to optics [1], a rigorous mathematical formulation of coherence as a quantum resource has been achieved only recently. In a remarkable paper, Baumgratz, Cramer and Plenio [2, 3] defined incoherent states as the states whose density matrix written in a chosen basis $\{|i\rangle\}$, do not have any off diagonal elements; $\rho = \sum_i p_i |i\rangle\langle i|$. Akin to entanglement not increasing under *local operation and classical communication (LOCC)* [4], incoherent operations have been defined as operations that map an incoherent state to an incoherent state. Two different coherence measures, relative entropy and l_1 norm have been proposed satisfying several monotonicity properties under incoherent operations [2]. Operational meaning of coherence in terms of the tasks coherence distillation and coherence cost under incoherent operation has been achieved [6]. Operational meaning of l_1 norm measure has also been explored extensively [5]. It has been shown that coherence is a necessary resource for generating entanglement using incoherent operations [7]. Quantum discord is another measure of correlations between two parties A and B, which quantifies difference between quantum mutual information $I(A : B)$ and classical mutual information $J(A : B)$ which is mutual information by doing a projective measurement on Alice or Bob's side [19, 20]. Discord is invariant under local unitary transformations and computing it is NP complete [21]. Interestingly, the upper bound on quantum discord that can be generated by incoherent operations has also been derived [8]. Distribution of coherence in multipartite system is explored and monogamy relations have been derived [10]. A basis independent measure of coherence for multipartite system has been proposed [9] and shown to be equal to the dis-

tance measure of quantum discord, based on relative entropy [13]. An unified approach of quantum correlations and coherence has been proposed [11]. A relative quantum coherence measure between two quantum states is defined and shown to be connected to various quantum correlation and non-locality without entanglement [12]. A scheme has been demonstrated for evaluating coherence experimentally for finite dimensional system [14]. Here, we provide an unified approach to quantum information theory and quantum correlations from the perspective of coherence. We consider the case, where Alice picks a state from the ensemble $\{\rho_1, \rho_2, \dots, \rho_n\}$ according to the probability distribution $\{p_1, p_2, \dots, p_n\}$ and sends it to Bob. Bob performs a projective measurement and gains some information about the state Alice has prepared. Holevo theorem sets the upper bound on the information gain of Bob. After performing a projective measurement, Bob always gets less or equal information than Holevo bound also known as quantum mutual information (QMI) [18]. We define the difference between QMI and information gain by a projective measurement as loss of information. Subsequently, we prove that the loss of information after performing a projective valued measurement (PVM) is identical to the loss of coherence with respect to the chosen measurement basis. Quantum discord [19, 20] for a bipartite system ρ_{AB} is the minimum difference (over all projectors) between the quantum mutual information $I(A : B)$ and classical mutual information $J(A : B)$. Here we prove that the difference between quantum mutual information and classical mutual information is equal to the difference between local coherence measure of subsystem B by local quantum incoherent operation and classical communication (LQICC) monotone and relative entropy measure of coherence of the partial system ρ_B . Partial density matrix ρ_B represents the scenario when Bob is completely ignorant about the subsystem A. Thus, $C_{rel.ent}(\rho_B)$ is the measure of local coherence available to Bob, when he completely ignores Alice, while the LQICC monotone measures the total coherence of subsystem B. Therefore discarding the subsystem A, decreases the coherence of B. It decreases even if A does not have any coherence which shows some coherence is stored in correlations.

In the following, we briefly review the concept of lo-

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cal coherence and a LQICC monotone based on relative entropy. Subsequently, we prove that the convexity property of the relative entropy measure of coherence can be derived from the Holevo bound, which implies that the loss of information due to mixing is equivalent to loss of coherence. We then obtain a relation between the discord and coherence measures.

II. A LOCAL COHERENCE MEASURE

The notion of local coherence for a bipartite density matrix ρ_{AB} has been recently introduced [16]. A bipartite density matrix σ_{AB} will have no local coherence or said to be *locally incoherent* on Bob side with respect to a chosen basis $\{|i\rangle\}$, if it can be written as,

$$\sigma_{AB} = \sum_k p_k \sigma_A^k \otimes \xi_B^k.$$

Here ξ_B^k is an incoherent state: $\xi_B^k = \sum_i p_i |i\rangle \langle i|$. The following measure of local coherence monotonous under LQICC operation has been proposed,

$$C_r^{A|B}(\rho_{AB}) = \min_{\sigma_{AB} \in QI} S(\rho_{AB} || \sigma_{AB}). \quad (1)$$

Here QI is the set of all locally incoherent states. The above *LQICC* monotone has been related to entropy [16],

$$C_r^{A|B}(\rho_{AB}) = S(\rho_{AB}^{d_B}) - S(\rho_{AB}),$$

where $\rho_{AB}^{d_B}$ is the block diagonal part of the density matrix ρ_{AB} in the chosen basis $\{|i\rangle\}$ for subsystem B.

III. HOLEVO BOUND AND COHERENCE

We consider the ensemble of quantum states $\{\rho_x\} = \{\rho_1, \rho_1, \dots, \rho_n\}$. Alice chooses a quantum state from this set, according to the probability distribution $\{p_x\} = \{p_1, p_2, \dots, p_n\}$ and sends it to Bob. Bob performs a *projective valued measurement (PVM)*, $E_Y = \{|i\rangle\}$ and tries to infer the state Alice has prepared. As is known, the accessible information to Bob $H(X : Y)$ is upper bounded by the Holevo's quantity, $\chi = S(\rho) - \sum_x p_x S(\rho_x)$; χ is also known as quantum mutual information [18]. The quantum state of the system for Bob reads, $\rho = \sum_x p_x \rho_x$. Bob performs the projective measurement $E_Y = \{|i\rangle\}$ on the state ρ and changes it into the post measurement state, $\rho^d = \sum_i \langle i | \sum_x p_x \rho_x | i \rangle | i \rangle \langle i |$, ρ^d is diagonal part of ρ in the basis $\{|i\rangle\}$. The information that Bob gains after performing a PVM, E_Y is, $H(Y) = S(\rho^d)$ and the average information gain if Bob knows the value of the random variable X, $H(Y|X) = \sum_x p_x S(\rho_x^d)$. Therefore, the mutual information of the variables X and Y reads,

$$H(X : Y) = H(Y) - H(Y|X) = S(\rho^d) - \sum_x p_x S(\rho_x^d),$$

Holevo bound, together with the above expression of mutual information leads to,

$$S(\rho^d) - \sum_x p_x S(\rho_x^d) \leq S(\rho) - \sum_x p_x S(\rho_x).$$

This is equivalent to the convexity property of coherence measured with respect to the basis $\{|i\rangle\}$:

$$C_{rel.ent}(\sum_x p_x \rho_x) \leq \sum_x p_x C_{rel.ent}(\rho_x). \quad (2)$$

One can define the difference between quantum mutual information χ and classical mutual information or accessible information $H(X : Y)$, as loss of information and the difference between average coherence of the ensemble $\sum_x p_x C_{rel.ent}(\rho_x)$ and the coherence of the mixed density matrix $C_{rel.ent}(\rho)$; $\rho = \sum_x p_x \rho_x$, as loss of coherence. Then the above shows the equivalence between these two quantities. Thus if loss of coherence is less in some basis, gain of information will be more by performing a measurement in that basis and if the loss of coherence of a quantum state sent over a quantum channel is zero in some basis, Holevo bound χ is achievable by choosing the same basis for measurement.

IV. QUANTUM DISCORD AND COHERENCE

For a bipartite quantum state ρ_{AB} , the mutual information between A and B reads,

$$I(A : B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$$

But after performing a measurement Bob can only retrieve $J(A : B)$ information about subsystem A,

$$J(A : B) = S(\rho_A) - S(\rho_{AB} | \{\Pi_i^B\})$$

The quantity $\delta(A : B)$ [19, 20] has been defined as the difference of quantum mutual information $I(A : B)$ and classical mutual information $J(A : B)$,

$$\delta(A : B) = I(A : B) - J(A : B)$$

or equivalently,

$$\delta(A : B) = S(\rho_B) - S(\rho_{AB}) + S(\rho_{AB} | \{\Pi_i^B\}).$$

and we also have [20],

$$J(A : B) \leq I(A : B)$$

If Bob chooses a projective measurement $\{|i\rangle_B \langle i|\}$; $S(\rho_{AB} | \{\Pi_i^B\}) = \sum_i p_i S(\rho_i)$. Here, $p_i = \text{tr}_{AB}(\rho_{AB} | i\rangle_B \langle i|)$ and $\rho_i = \text{tr}_B(\rho_{AB} | i\rangle_B \langle i|) \otimes |i\rangle \langle i| / p_i$. Quantum discord $D(A : B)$ is the minimum value of $\delta(A : B)$ over all possible projective measurements $\{\{\Pi_j^B\}\}$ at Bob's side. It measures the mutual information of subsystems A and B that is inaccessible by

performing a local measurement on Bob's side. Using the expression [17],

$$S(\sum_i p_i \rho_i) = S(\rho_{AB}^d) = H(p_i) + \sum_i p_i S(\rho_i),$$

one obtains,

$$\delta(A : B) = S(\rho_B) - S(\rho_{AB}) + S(\rho_{AB}^d) - H(p_i).$$

Since $H(p_i) = S(\rho_B^d)$, we have the following relation,

$$\delta(A : B) = C_r^{(A|B)}(\rho_{AB}) - C_{rel.ent}(\rho_B), \quad (3)$$

and also,

$$D(A : B) = \min_{\{|i\rangle\langle i|\}} (C_r^{(A|B)}(\rho_{AB}) - C_{rel.ent}(\rho_B)). \quad (4)$$

The positivity of quantum discord follows from the following inequality,

$$S(\rho_{AB} || \rho_{AB}^d) \geq S(\rho_B || \rho_B^d).$$

$Eq[3]$ shows that the difference between quantum mutual information and classical mutual information is equal to the difference between local coherence on Bob's side given by the LQICC monotone and relative entropy measure of coherence of the reduced density matrix ρ_B . $C_{rel.ent}(\rho_B)$ measures the local coherence of Bob when he is not aware about Alice's system, while the LQICC monotone measures the total coherence of subsystem

B . Thus, the difference represents the coherence loss of subsystem B , when it ignores A . It is counter-intuitive that the coherence of subsystem B can decrease after discarding the system A even if subsystem A does not have any coherence. As an example, consider density matrix $\rho_{AB} = \frac{1}{2} |0\rangle\langle 0| \otimes |0\rangle\langle 0| + \frac{1}{2} |1\rangle\langle 1| \otimes |+\rangle\langle +|$, LQICC monotone for subsystem A is zero, as also the relative entropy of coherence for the partial density matrix ρ_A with respect to the basis $\{|0\rangle, |1\rangle\}$, however $C_r^{A|B}(\rho_{AB}) - C_{rel.ent}(\rho_B) \approx 0.289$, with respect to the basis $\{|0\rangle, |1\rangle\}$. Hence the lost coherence is hidden in correlations between subsystem A and B .

To conclude, we have considered single particle and bipartite quantum state and demonstrated the equivalence between loss of information and loss of quantum coherence. In single particle case, we established an exact relationship between Holevo bound and the convexity property of coherence which implies that the loss of information is equivalent to the loss of coherence. For the bipartite case, we proved the equivalence between quantum discord and locally inaccessible coherence.

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